THEORY OF COMPRESSIVE STRESS IN ALUMINUM OF ACSR

Nomenclature

$A_a =$ total aluminum area
$A_i =$ area of strand in ith layer
$d =$ strand diameter
$D =$ outside diameter of strand layer
$E =$ Young's modulus
$l =$ length of wire in one lay length
$n_i =$ number of strands in ith layer
$P =$ Total tension or compression load in aluminum part of ACSR
$R =$ radius of strand helix (radius from conductor axis to strand axes)
$T =$ tension (or compression) force in one strand of ith layer
$Y =$ interlayer pressure per unit length of strand
$\alpha =$ lay angle of strand helix at strand axis
$\Delta_a =$ Radial expansion of aluminum portion as a unit
$\epsilon =$ conductor axial strain
$\epsilon_n =$ conductor strain during unloading after aluminum goes slack
$\kappa =$ curvature of strand due to helicity $= \sin^2\alpha/R$
$\lambda =$ lay length

Introduction

Consider an ACSR that has been subjected to some initial loading, following which the load is reduced. The unloading will follow the final stress-strain curve, and at some point the tension in the aluminum portion will go to zero. Since the aluminum will have experienced some plastic deformation, while the steel will have experienced little if any, the steel will still be under tension when the aluminum "goes slack."

As the conductor tension is reduced further, the aluminum layers will tend to expand away from the core and become loose. The rates at which the individual aluminum layers expand will usually be different, due to differences in the layer diameters and the lay angles of the strands. A shallow lay angle or large lay ratio will cause more rapid radial expansion than a large lay angle or short lay ratio. Because of this, there are conditions where an inner layer will expand radially more rapidly than the layer above it, and there will be interference between them. When that happens, the layers must press against and strain against each other, and one effect of this is a net compressive stress developing in the pair of layers taken as a unit. If there are more than two aluminum layers, then three or more layers may lock together in this manner, depending upon their relative rates of radial expansion following unloading of the conductor past the "slack aluminum" point.

The effect of this net compressive stress is to extend the final stress-strain curve for the aluminum into the *negative* stress range, rather than simply having it stop at zero stress as is generally assumed. In that range, the effective aluminum modulus is different from that in the positive stress range, being determined by the degree to which the various layers interfere with each others' expansion. We can calculate this negative-stress leg of the final aluminum curve by analyzing that interference.

Analysis

We will focus on what happens as conductor elongation is further reduced, after having reached the point where the aluminum goes slack. Let us assume that all aluminum layers go slack simultaneously, although this is probably only approximately true. We will begin the analysis by determining how rapidly each layer would expand radially, during this further reduction in elongation, in the absence of interference. Then we will determine how much each layer must be deflected radially from its free expansion position in order to eliminate the interference, and yet leave the locked-together set of interfering layers in mechanical equilibrium. These individual deflections will require strains in the strands of the layers in question, and those strains will determine the layer tensions. The sum of these tensions is the tension in the aluminum, and that may be used to calculate an overall average aluminum stress. The ratio of this stress to the conductor strain that provoked the interference between layers is the aluminum modulus in the negative stress range.

Let the decreasing conductor strain that follows the point where the aluminum goes slack be ϵ_n. Now the length of strand in one lay length is,

$\cos\alpha = \frac{\lambda}{l}$

$$l = \sqrt{\lambda^2 + 4\pi^2 \cdot R^2}\ , \quad \text{so} \quad R = \frac{\sqrt{l^2-\lambda^2}}{2\pi}$$

$\tan\alpha = \sqrt{\left(\frac{l}{\lambda}\right)^2 - 1} = \frac{2\pi R}{\lambda}$

If there is no interference between layers when ϵ_n occurs, the strand will experience negligible stress, so l will be constant. Lay length λ will change, however, because of ϵ_n, and the rate at which R varies with λ is,

$$\frac{\partial R}{\partial \lambda} = -\frac{2\cdot\lambda}{4\pi\cdot\sqrt{l^2-\lambda^2}} = -\frac{1}{2\pi\cdot\tan\alpha}$$

But the conductor strain is $\epsilon_n = \frac{d\lambda}{\lambda}$, so $d\lambda = \lambda\cdot\epsilon_n$, and the radial expansion of the layer is,

$$\Delta R = -\frac{\lambda}{2\pi\cdot\tan\alpha}\cdot\epsilon_n$$

Note that ϵ_n is negative, so ΔR is positive. Now, if the outside diameter of the layer is D and strand diameter is d, then

$R = \frac{1}{2}(D-d)$ → $$\tan\alpha = \frac{\pi\cdot(D-d)}{\lambda}$$

But in a well-packed strand layer, $D-d = \frac{n\cdot d}{3}$, so $\lambda = \frac{\pi\cdot n\cdot d}{3\cdot\tan\alpha}$ and,

$R = \frac{nd}{6}$ $$\Delta R = -\frac{n\cdot d}{6\cdot\tan^2\alpha}\cdot\epsilon_n$$

If we identify the individual layers by the subscript, $i = 1, 2, 3, \ldots$, then

$$\Delta R_i = -\frac{n_i\cdot d_i}{6\cdot\tan^2\alpha}\cdot\epsilon_n \tag{1}$$

These ΔR_i are generally different, even though all layers experience the same *conductor* strain, ϵ_n. The next step in our analysis is to force all interfering layers to share the same radial deflection, Δ_a. This will require additional deflections, δR_i, such that,

$$\Delta R_i + \delta R_i = \Delta_a \tag{2}$$

for all interfering layers. In reality, these additional deflections take place simultaneously with the ΔR_i, but it is convenient for us to treat them as though they took place sequentially. Thus, we have treated the ΔR_i as though they took place without change in strand tension, letting that remain zero. The layers expanded without restraining each other. Now we will treat the δR_i as occurring in the absence of additional conductor strain, as we force the layers to the position where they are in contact, but do not interpenetrate.

If the conductor elongation is indeed held constant during this second step, then so is lay length, λ, so the rate of change of helix radius with respect to wire length is,

$$\frac{\partial R}{\partial l} = \frac{1}{2\pi}\cdot\frac{2\cdot l}{2\cdot\sqrt{l^2-\lambda^2}} = \frac{1}{2\pi}\cdot\frac{1}{\sqrt{1-\frac{\lambda^2}{l^2}}} = \frac{1}{2\pi\cdot\sin\alpha}$$

So,

$$\delta R = \frac{l}{2\pi \cdot \sin\alpha} \cdot \frac{dl}{l}$$

But, $\sin\alpha = \frac{\pi\cdot(D-d)}{l}$, so $l = \frac{\pi\cdot(D-d)}{\sin\alpha}$. Also, $D-d = \frac{n\cdot d}{3}$, so,

$$\delta R_i = \frac{n_i\cdot d_i}{6\cdot\sin^2\alpha_i}\cdot\left(\frac{dl}{l}\right)_i \tag{3}$$

This gives the radial deflection of the layer as a function of the longitudinal strain of the wire of the strands, dl/l. We will take the source of dl/l to be a change in strand tension, T_i, and this tension arises because the interfering layers press against each other.

If the area of a strand in the ith layer is A_i, then

$$T_i = E_i A_i\cdot\left(\frac{dl}{l}\right)_i, \quad \text{so that} \quad \delta R_i = \frac{n_i\cdot d_i}{6\cdot E_i A_i\cdot\sin^2\alpha_i}\cdot T_i$$

The tension required to cause deflection δR_i is thus,

$$T_i = \frac{6\cdot E_i A_i\cdot\sin^2\alpha_i}{n_i d_i}\cdot\delta R_i \tag{4}$$

Now, strand tension ordinarily results in a binding pressure from the layer in question upon the layer below. In the present case, however, the inner layer(s) of the interfering group of layers will be in compression, so they will exert a pressure outward to meet the inward pressures from above. Within the group of layers that is expanding as a unit, the radial forces from the various layers must be in equilibrium; they must add up to zero, since the group is out of contact with layers below and above. We need to relate these interlayer forces to the strand tensions, T_i.

Let the radial force acting between the layers be Y per unit strand length for each strand in the layer. Now it can be shown that $Y = \kappa\cdot T$, where T is the tension (or compression) in the strand along its axis, and κ is the curvature of the strand. For a helix, $\kappa = \sin^2\alpha/R$. The total inward radial force from a layer, per unit length along the *conductor* is thus,

$$\frac{n\cdot Y}{\cos\alpha} = \frac{n}{\cos\alpha}\cdot\frac{\sin^2\alpha}{R}\cdot T = n\cdot\sin\alpha\cdot\tan\alpha\cdot\frac{6}{nd}\cdot T = \frac{6}{d}\cdot\sin\alpha\cdot\tan\alpha\cdot T$$

The cos α in the denominator occurs because we are now working per unit length of conductor, instead of strand. These interlayer radial forces must add up to zero, so

$$6 \cdot \sum_i \frac{\sin\alpha_i \cdot \tan\alpha_i}{d_i} \cdot T_i = 0 \tag{5}$$

Substituing (4) into (5), ,

$$6 \cdot \sum_i \frac{\sin\alpha_i \cdot \tan\alpha_i}{d_i} \cdot \frac{6 \cdot E_i A_i \cdot \sin^2\alpha_i}{n_i d_i} \cdot \delta R_i = 0$$

and,

$$\sum_i \frac{E_i A_i}{n_i d_i^2} \cdot \sin^3\alpha_i \cdot \tan\alpha_i \cdot \delta R_i = 0 \tag{6}$$

This gives one equation relating the δR_i as unknowns. All other parameters in (6) are defined by the conductor structure. The equation imposes balance of radial forces within the interfering group of layers.

Equation (1) may be substituted into (2), to eliminate the ΔR_i thus:

$$\delta R_i = \Delta_a + \frac{n_i \cdot d_i}{6 \cdot \tan^2\alpha_i} \cdot \epsilon_n \tag{7}$$

providing as many equations as there are interfering layers. These equations, with (6) form a set of simultaneous equations that maybe solved for Δ_a and the δR_i as unknowns.

For compactness, define

$$C_i = \frac{E_i A_i}{n_i d_i^2} \cdot \sin^3\alpha_i \cdot \tan\alpha_i \tag{8}$$

$$= \frac{\pi E_i}{4 n_i} \cdot \sin^3\alpha_i \cdot \tan\alpha_i$$

and

$$B_i = \frac{n_i \cdot d_i}{6 \cdot \tan^2\alpha_i} \tag{9}$$

Then these equations can be written in matrix form. For illustration we will assume three interfering layers. Thus,

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ C_1 & C_2 & C_3 & 0 \end{bmatrix} \cdot \begin{Bmatrix} \delta R_1 \\ \delta R_2 \\ \delta R_3 \\ \Delta_a \end{Bmatrix} = \epsilon_n \cdot \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ 0 \end{Bmatrix} \tag{10}$$

$$\begin{bmatrix} \delta R_1 \\ \delta R_2 \\ \delta R_3 \\ \Delta_a \end{bmatrix} = -\epsilon_n \begin{bmatrix} \frac{B_2C_2+B_3C_3-B_1(C_2+C_3)}{C_1+C_2+C_3} \\ \frac{B_1C_1+B_3C_3-B_2(C_1+C_3)}{C_1+C_2+C_3} \\ \frac{B_1C_1+B_2C_2-B_3(C_1+C_2)}{C_1+C_2+C_3} \\ \frac{B_1C_1+B_2C_2+B_3C_3}{C_1+C_2+C_3} \end{bmatrix}$$

This may be written even more compactly as,

$$C \cdot \delta = \epsilon_n \cdot B \qquad (11)$$

where C is the square matrix, δ is the column vector, $\{\delta R_1 \;\; \delta R_2 \;\; \delta R_3 \;\; \Delta_a\}^T$, and B is the column vector, $\{B_1 \; B_2 \; B_3 \; 0\}^T$.

Then,

$$\delta = \epsilon_n \cdot C^{-1} \cdot B \qquad (12)$$

where C^{-1} is the matrix inverse of C. Now, from (4), the tension per strand in the ith layer is,

$$T_i = \frac{6 \cdot E_i A_i \cdot \sin^2 \alpha_i}{n_i d_i} \cdot \delta R_i$$

Define

$$H_i = \frac{6 \cdot E_i A_i \cdot \sin^2 \alpha_i}{n_i d_i} \qquad (13)$$

Then

$$T_i = H_i \cdot \delta R_i \qquad (14)$$

The component of T_i in the direction of the conductor axis is $T_i \cdot \cos \alpha_i$, so the contribution of that layer to conductor tension is,

$$n_i T_i \cdot \cos \alpha_i = n_i H_i \cdot \cos \alpha_i \cdot \delta R_i$$

Thus, the total tension in the group of interfering layers is,

$$P = \sum_i n_i H_i \cdot \cos \alpha_i \cdot \delta R_i \qquad (15)$$

Define the row vector, $F = \{n_1 H_1 \cos \alpha_1 \quad n_2 H_2 \cos \alpha_2 \quad n_3 H_3 \cos \alpha_3 \; ... \; 0\}$
Then P is given by a quadratic form:

$$P = \epsilon_n \cdot F \cdot C^{-1} \cdot B \qquad (16)$$

Let the total aluminum area be A. Then the effective aluminum modulus in the negative stress region is,

$$E_n = \frac{P}{A \cdot \epsilon_n} = \frac{1}{A} \cdot F \cdot C^{-1} \cdot B \qquad (17)$$

Note that E_n is defined on the aluminum area, rather than the total conductor area, so it must be multiplied by H_a before being used in sag-tension calculations.

Cautions

This analysis assumes that the magnitude of Δ_a is small compared to the layer radii. In addition, it is assumed that the the expansion of the aluminum does not become localized, forming a birdcage. Rather, the looseness of the aluminum is taken to be uniform along the conductor. There is evidence from Nigol *et al* that, at least at high conductor temperatures, the buckling of the aluminum may concentrate into localized birdcages. In that case, the effective value of E_n is reduced. Finally, it is assumed that the normal compliance at interlayer contacts is negligible.

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Addendum

The equation between (4) and (5) is not complete. The radial pressure Y is influenced not only by T but also by the bending moment M and torque H in the strand. The complete relation is, from Love, §254, Eqs (10) and (11) as applied to a uniform helix,

$$Y = T\kappa - H\kappa\tau + M\tau^2 \tag{A1}$$

We consider the helix to be in equilibrium before being deflected by δR, so we are concerned with increments in T, H and M, with κ and τ sensibly constant. Thus,

$$\frac{\delta Y}{\delta R} = \kappa\frac{dT}{dR} - \kappa\tau\frac{dH}{dR} + \tau^2\frac{dM}{dR} \tag{A2}$$

Now,

$$\frac{dT}{dR} = \frac{dT}{d\varepsilon}\frac{d\varepsilon}{dR} = EA\frac{1}{l}\frac{dl}{dR} \tag{A3}$$

$$\frac{dH}{dR} = \frac{dH}{d\tau}\frac{d\tau}{dR} = G\Phi\frac{d\tau}{dR} \tag{A4}$$

$$\frac{dM}{dR} = \frac{dM}{d\kappa}\frac{d\kappa}{dR} = EI\frac{d\kappa}{dR} \tag{A5}$$

Also,

$$l = \sqrt{\lambda^2 + 4\pi^2R^2} \qquad \sin\alpha = \frac{2\pi R}{\sqrt{\lambda^2 + 4\pi^2R^2}} \qquad \cos\alpha = \frac{\lambda}{\sqrt{\lambda^2 + 4\pi^2R^2}} \tag{A6}$$

$$\kappa = \frac{\sin^2\alpha}{R} = \frac{4\pi^2 R}{\lambda^2 + 4\pi^2R^2} \tag{A7}$$

$$\tau = \frac{\sin\alpha\cos\alpha}{R} = \frac{2\pi\lambda}{\lambda^2 + 4\pi^2R^2} \tag{A8}$$

Then,

$$\frac{dl}{dR} = \frac{4\pi^2 R}{\sqrt{\lambda^2 + 4\pi^2R^2}} = 2\pi\sin\alpha$$

and

$$\frac{1}{l}\frac{dl}{dR} = \frac{2\pi\sin^2\alpha}{R} \tag{A9}$$

Furthermore,

$$\frac{d\kappa}{dR} = \frac{\sin^2\alpha}{R^2} \cdot \left(1 - 2\sin^2\alpha\right) \tag{A10}$$

and,

$$\frac{d\tau}{dR} = -2\,\frac{\sin^3\alpha \cdot \cos\alpha}{R^2} \tag{A11}$$

Collecting all parts,

$$\frac{dY}{dR} = EA\,\frac{\sin^4\alpha}{R^2} - 2G\Phi\,\frac{\sin^6\alpha\cdot\cos^2\alpha}{R^4} + EI\,\frac{\sin^4\alpha\cdot\cos^2\alpha}{R^4}\left(1 - 2\sin^2\alpha\right) \tag{A12}$$

Since $A = \pi d^2/4$, $I = \pi d^4/64$, and $G = 2I$, for circular strands,

$$\frac{dY}{dR} = E\,\frac{\pi}{4}\left(\frac{d}{R}\right)^2 \sin^4\alpha + E\,\frac{\pi}{64}\left(\frac{d}{R}\right)^4 \sin^4\alpha\,\cos^2\alpha$$

$$-\frac{\pi}{32}\,(2G + E)\left(\frac{d}{R}\right)^4 \sin^6\alpha\,\cos^2\alpha \tag{A13}$$

R = nd/6

However, since $R = 6/nd$ in a well packed layer,

$$\frac{dY}{dR} = E\,\frac{9\pi}{n^2}\sin^4\alpha + E\,\frac{81\pi}{4n^4}\sin^4\alpha\,\cos^2\alpha - (2G + E)\frac{81\pi}{2n^4}\sin^6\alpha\,\cos^2\alpha \tag{A14}$$

The total force, P, per unit length of conductor from all strands in the layer will be $n/\cos\alpha$ times this. This results in,

$$P = E\,\frac{9\pi}{n}\sin^3\alpha\tan\alpha\left[1 + \frac{9}{4n^2}\cos^2\alpha - \left(1 + 2\frac{G}{E}\right)\frac{9}{2n^2}\sin^2\alpha\,\cos^2\alpha\right] \tag{A15}$$

Thus, the C_i in (8) should be multiplied by the factor in brackets to obtain a more nearly correct value of E_n.

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$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ C_1 & C_2 & C_3 & 0 \end{bmatrix}^{-1} = \frac{1}{(C_3+C_2+C_1)} \cdot \begin{bmatrix} C_3+C_2 & -C_2 & -C_3 & 1 \\ -C_1 & C_3+C_1 & -C_3 & 1 \\ -C_1 & -C_2 & C_2+C_1 & 1 \\ -C_1 & -C_2 & -C_3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ C_1 & C_2 & C_3 & C_4 & 0 \end{bmatrix}^{-1} =$$

$$\frac{1}{(C_4+C_3+C_2+C_1)} \cdot \begin{bmatrix} C_4+C_3+C_2 & -C_2 & -C_3 & -C_4 & 1 \\ -C_1 & C_4+C_3+C_1 & -C_3 & -C_4 & 1 \\ -C_1 & -C_2 & C_4+C_2+C_1 & -C_4 & 1 \\ -C_1 & -C_2 & -C_3 & C_3-C_2+C_1 & 1 \\ -C_1 & -C_2 & -C_3 & -C_4 & 1 \end{bmatrix}$$

Compression Modulus of a Free Strand Layer

When the aluminum layers of an ACSR go slack, following a tension loading that leaves permanent set in them, they are able to sustain some compressive load because they act as helical compression springs. The stiffness, or spring constant for a helical strand can be calculated from Love[1]. This equation actually gives the axial force and torque that result from axial deflection δh and torsional deflection $\delta \chi$. Since we are not interested in the change in torque, and are assuming that the conductor does not twist, we need only the part of (42) that gives,

$$R = \frac{1}{l\,r^2}\left(C \cdot \cos^2\theta + B \cdot \sin^2\theta\right)\delta h \tag{1}$$

where C is the torsional rigidity of the strand or wire and B is its flexural rigidity. (As θ approaches zero, this equation approaches the formula for the constant of a coiled compression spring as given by Marks.[2]) For round wire,

$$B = EI = E \cdot \frac{\pi\, d^4}{64}\,, \text{ and } \; C = G\Phi = \frac{E}{2\,(1+\nu)} \cdot \frac{\pi\, d^4}{32} \tag{2}$$

Thus,

$$\frac{R}{\delta h} = \frac{1}{l\,r^2}\left(\frac{\cos^2\theta}{1+\nu} + \sin^2\theta\right) \cdot E \cdot \frac{\pi\, d^4}{64} \tag{3}$$

Now, per unit length of conductor, $l = 1/\sin\theta$. Furthermore, in a well packed strand layer, $r = nd/6$. Thus, the spring constant for a strand becomes,

$$\frac{R}{\delta h} = \frac{36\sin\theta}{n^2 d^2} \cdot E \cdot \frac{\pi\, d^4}{64} \cdot \left(\frac{\cos^2\theta}{1+\nu} + \sin^2\theta\right)$$

$$= \frac{9\pi\, d^2 E \sin\theta}{16\, n^2} \cdot \left(\frac{\cos^2\theta}{1+\nu} + \sin^2\theta\right)$$

The spring constant per unit area of strand, that is, its apparent Young's Modulus, is.

$$E_{eff} = \frac{4}{\pi d^2} \cdot \frac{R}{\delta h} \;=\; \frac{9}{4} \cdot \left(\frac{\cos^2\theta}{1+\nu} + \sin^2\theta\right) \cdot \frac{E}{n^2} \sin\theta \tag{4}$$

Note that Love's $\theta = \pi/2 - \alpha$, where α is the angle between the helix axis and the strand axis. The factor,

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[1] *The Mathematical Theory of Elasticity*, by A. E. H. Love, Dover, 1944, §271, Eq. (42).

[2] *Marks' Engineers' Handbook*, 4th Edition, page 486, ¶15.

$$\frac{9}{4} \cdot \left(\frac{\cos^2\theta}{1+\nu} + \sin^2\theta \right) \cdot \sin\theta$$

is nearly constant for practical values of α. Taking $\nu = 1/3$,

α	5°	10°	15°
Factor	2.237	2.199	2.137

Thus, to a good approximation,

$$E_{eff} \approx 2.2 \cdot \frac{E}{n^2} \quad (5)$$

There is actually some additional compliance of the strand that results from the compressive stress on it. From Progress Report 16-P-77, Eq (10a),

$$\frac{\partial T}{\partial \varepsilon} = EA\cos^3\alpha \quad (6)$$

The effective modulus then becomes,

$$E_{eff} = \frac{1}{\frac{1}{\frac{9}{4}\cdot\left(\frac{\cos^2\theta}{1+\nu}+\sin^2\theta\right)\cdot\frac{E}{n^2}\sin\theta} + \frac{1}{E\sin^3\theta}} \quad (7)$$

$$\approx \frac{1}{\frac{1}{2.2\cdot\frac{E}{n^2}} + \frac{1}{E\cos^3\alpha}} = \frac{E}{\frac{n^2}{2.2} + \frac{1}{\cos^3\alpha}} \quad (8)$$

For a 6 strand layer with $\alpha = 15°$, this changes E_{eff} from 611,111 psi to 572,304, a reduction of 6.35%. For a 7 strand layer, the change is 4.74%; for 8 strand, 3.00%; 9 strand, 2.92%; 10 strand, 2.38%.
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